

# On Existences Of Periodic Orbits For Hamilton Systems\*

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## Abstract

In this article, we prove that either there exists at least one periodic orbit of Hamilton vector field on a given energy hypersurface in  $R^{2n}$  or there exist at least two periodic orbits on the near-by energy hypersurface in  $R^{2n}$ . The more general results are also obtained.

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## 1 Introduction and results

Let  $\Sigma$  be a smooth closed oriented manifold of dimension  $2n - 1$  in  $R^{2n}$ , here  $(R^{2n}, \omega_0)$  ( $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ ) is the standard symplectic space. Then there exists a unique vectorfield  $X_\Sigma$  so called Hamilton vector field defined by  $i_{X_\Sigma} \omega_0|_\Sigma \equiv 0$ . A periodic Hamilton orbit in  $\Sigma$  is a smooth path  $x : [0, T] \rightarrow \Sigma$ ,  $T > 0$  with  $\dot{x}(t) = X_\Sigma(x(t))$  for  $t \in (0, T)$  and  $x(0) = x(T)$ . Seifert in [22] raised the following conjecture:

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**Conjecture**(see[22]). Let  $\omega_0$  be the standard symplectic form on  $R^{2n}$ . Let  $\Sigma$  be a closed  $(2n - 1)$ -hypersurface in  $R^{2n}$ . Then there is a closed Hamilton periodic orbit in  $\Sigma$ .

Rabinowitz [21] and Weinstein [25, 26] proved that if  $S$  is starshape resp. convex,  $SC$  holds. Weinstein conjectured  $SC$  holds for the hypersurface of contact type in general symplectic manifold( $WC$ ). In  $R^{2n}$ ,  $SC$  implies  $WC$ . Viterbo [24] proved the  $WC$  in  $R^{2n}$ . By the Viterbo's work, Hofer and Zehnder in [13] proved the near-by  $SC$  holds. Struwe in [23] proved that  $SC$  almost holds by modifying Hofer-Zehnder's work. Ginzburg in [7] and Herman in [10] gave a counter-example [10] for the  $SC$ . After Viterbo's work, many results were obtained by using variational method or Gromov's  $J$ -holomorphic curves via nonlinear Fredholm alternative, see [5, 11, 12, 13, 15, 16, 17, 18] etc.

Let  $(M, \omega)$  be a symplectic manifold and  $h(t, x)(= h_t(x))$  a compactly supported smooth function on  $M \times [0, 1]$ . Assume that the segment  $[0, 1]$  is endowed with time coordinate  $t$ . For every function  $h$  define the (*time - dependent*) *Hamiltonian vector field*  $X_{h_t}$  by the equation:

$$dh_t(\eta) = \omega(\eta, X_{h_t}) \text{ for every } \eta \in TM \quad (1.1)$$

The flow  $g_h^t$  generated by the field  $X_{h_t}$  is called *Hamiltonian flow* and its time one map  $g_h^1$  is called *Hamiltonian diffeomorphism*.

Now assume that  $H$  be a time independent smooth function on  $M$  and  $X_H$  its induced vector field.

Let  $(M, \omega)$  be a symplectic manifold. Let  $J$  be the almost complex structure tamed by  $\omega$ , i.e.,  $\omega(v, Jv) > 0$  for  $v \in TM$ . Let  $\mathcal{J}$  the space of all tame almost complex structures.

**Definition 1.1** *Let*

$$s(M, \omega, J) = \inf \left\{ \int_{S^2} f^* \omega > 0 \mid f : S^2 \rightarrow M \text{ is } J\text{-holomorphic} \right\}$$

**Definition 1.2** *Let*

$$s(M, \omega) = \sup_{J \in \mathcal{J}} l(M, \omega, J)$$

Let  $W$  be a Lagrangian submanifold in  $M$ , i.e.,  $\omega|_W = 0$ .

**Definition 1.3** *Let*

$$l(M, W, \omega) = \inf\{|\int_{D^2} f^* \omega| > 0 | f : (D^2, \partial D^2) \rightarrow (M, W)\}$$

**Theorem 1.1** *Let  $(M, \omega)$  be a closed compact symplectic manifold or a manifold convex at infinity and  $M \times C$  be a symplectic manifold with symplectic form  $\omega \oplus \sigma$ , here  $(C, \sigma)$  standard symplectic plane. Let  $2\pi r_0^2 < s(M, \omega)$  and  $B_{r_0}(0) \subset C$  the closed ball with radius  $r_0$ . Assume that  $H$  be a time independent smooth function on  $M \times C$  and  $X_H$  its induced vector field. If  $\Sigma = H^{-1}(c)$  be a smooth hypersurface in  $M \times B_{r_0}(0)$ ,  $X_H$  its Hamilton vector field, then either there exists at least one periodic orbits of  $X_H$  on  $\Sigma$  or there exists at least two periodic orbits of  $X_H$  on  $\Sigma' = H^{-1}(c')$ ,  $c'$  is close to  $c$  as one wants.*

**Theorem 1.2** *Let  $(M, \omega)$  be an exact symplectic manifold, i.e.,  $\omega = d\alpha$  for some 1-form  $\alpha$ . Assume that  $H$  be a time independent smooth function on  $M$  and  $X_H$  its induced vector field. If  $\Sigma = H^{-1}(c)$  be a smooth compact hypersurface in  $M$  and there exists a Hamiltonian diffeomorphism  $h$  such that  $h(\Sigma) \cap \Sigma = \emptyset$ , then either there exists at least one periodic orbits of  $X_H$  on  $\Sigma$  or there exists at least two periodic orbits of  $X_H$  on  $\Sigma' = H^{-1}(c')$ ,  $c'$  is close to  $c$  as one wants.*

The near-by and almost existence results or the finity of Hofer-Zehnder's for bounded set in the above symplectic manifolds can be obtained as in [14]. For example, one has

**Corollary 1.1** *Let  $M$  be any open manifold and  $(T^*M, d\alpha)$  be its cotangent bundle. Assume that  $B$  is bounded set in  $T^*M$ , then the Hofer-Zehnder capacity  $C_{HZ}(B)$  is finite.*

Theorem 1.1-1.2 was reported in the proceedings of the international conference on "Boundary Value Problems, Integral Equations, And Related Problems" (5-13 August 2002); "ICM2002-Beijing Satellite Conference on Nonlinear Functional Analysis, August 14-18, 2002 Taiyuan. Theorem 1.1-1.2 can be generalized to the products of symplectic manifolds in Theorem 1.1-1.2. The proofs of Theorem 1.1-1.2 is close as in [17, 19]. Here we flow the Monke's method in [19]. If a  $(n-1)$ -dimensional submanifold  $\mathcal{L}$  in  $\Sigma$  satisfying that  $\mathcal{L}$  is transversal to the hamilton vector field  $X_H$  and  $\omega_0|_{\mathcal{L}} = 0$  and

$\omega_0|_{\pi_2(M, \mathcal{L})} = 0$ , then we call  $\mathcal{L}$  the Hamilton-Legendre isotropic submanifold. A Hamilton-Arnold chord in  $\Sigma$  is a smooth path  $x : [0, T] \rightarrow \Sigma, T > 0$  with  $\dot{x}(t) = X_\Sigma(x(t))$  for  $t \in (0, T)$  and  $x(0), x(T) \in \mathcal{L}$ . Then one can also prove the Hamilton's chord almost existence results as Theorem 1.1-1.2.

## 2 Lagrangian Non-Squeezing

**Theorem 2.1** ([20]) *Let  $(M, \omega)$  be a closed compact symplectic manifold or a manifold convex at infinity and  $M \times C$  be a symplectic manifold with symplectic form  $\omega \oplus \sigma$ , here  $(C, \sigma)$  standard symplectic plane. Let  $2\pi r_0^2 < s(M, \omega)$  and  $B_{r_0}(0) \subset C$  the closed disk with radius  $r_0$ . If  $W$  is a close Lagrangian manifold in  $M \times B_{r_0}(0)$ , then*

$$l(M, W, \omega) < 2\pi r_0^2$$

This can be considered as an Lagrangian version of Gromov's symplectic non-squeezing([8]).

**Corollary 2.1** (Gromov[8]) *Let  $(V', \omega')$  be an exact symplectic manifold with restricted contact boundary and  $\omega' = d\alpha'$ . Let  $V' \times C$  be a symplectic manifold with symplectic form  $\omega' \oplus \sigma = d\alpha = d(\alpha' \oplus \alpha_0)$ , here  $(C, \sigma)$  standard symplectic plane. If  $W$  is a close exact Lagrangian submanifold, then  $l(V' \times C, W, \omega) = \infty$ , i.e., there does not exist any close exact Lagrangian submanifold in  $V' \times C$ .*

**Corollary 2.2** *Let  $L^n$  be a close Lagrangian in  $R^{2n}$  and  $L(R^{2n}, L^n, \omega) = 2\pi r_0^2 > 0$ , then  $L^n$  can not be embedded in  $B_{r_0}(0)$  as a Lagrangian submanifold.*

## 3 Construction Of Lagrangian

### 3.1 First Case: no periodic orbit

Let  $(V, \omega) = (R^{2n} \times R^{2n}, \omega_0 \ominus \omega_0)$  be the standard symplectic vector space, here  $\omega_0 = d\lambda_0 = d(\frac{1}{2}(x_i dy_i - y_i dx_i))$ . Let  $\Sigma$  be a oriented closed hypersurface in  $R^{2n}$ . Let  $\mathcal{L} = \{(\sigma, \sigma) | \sigma \in \Sigma \subset R^{2n}\}$  be a closed isotropic submanifold contained in  $(\Sigma', \omega) = (\Sigma \times \Sigma, \omega_0 \ominus \omega_0)$ , i.e.,  $Q^*\omega|_{\mathcal{L}} = 0$ . Since  $\Sigma$  is oriented

in  $R^{2n}$ , the normal bundle of  $\Sigma$  is trivial. So, the tubular neighbourhood  $Q_\delta(\Sigma)$  of  $\Sigma$  is foliated by  $\Sigma$ . We now define a Hamiltonian as follows. Define  $Q_\delta(\Sigma) = \cup_{|t| \leq \delta} \psi_t(\Sigma)$ , here  $\psi_t$  is the flow of the normal vector field of  $\Sigma$ . Define  $h(x) = t(x)$ ,  $t(x)$  is the arrival time of  $\psi_t$  from  $x$  to  $\Sigma$  for  $x \in Q_\delta(\Sigma)$ . Let  $H(\sigma_1, \sigma_2) = h(\sigma_2)$  on  $\bar{Q}_\delta = Q_\delta(\Sigma) \times Q_\delta(\Sigma)$  and  $X_H$  be its Hamilton vectorfield. Let  $\eta_s$  be the Hamilton flow on  $\bar{Q}_\delta$  induced by  $X_H$ . Let  $S$  be a smooth compact hypersurface in  $R^{4n}$  which intersects the hypersurface  $E = R^{2n} \times \Sigma$  transversally and contains  $\mathcal{L}$ . Furthermore,  $S$  is transversal to the hamilton vector field  $X_H$ . Let  $s(x)$  be the time to  $S$  of hamilton flow  $\eta_s$  which is well defined in the neighbourhood  $U_{\delta_1}(S)$ . Let  $X_S$  be the Hamilton vector field of  $S$  on  $U_{\delta_1}$  and  $\xi_t$  be the flow of  $X_S$  defined on  $U_{\delta_1}$ . Since  $X_S(H) = \{s, H\} = -\{H, s\} = -X_H(s) = -1 \neq 0$ , here  $\{, \}$  is the Poisson bracket, so  $X_S$  is transversal to  $E$ . Then, there exists  $\delta_0$  such that  $\xi_t(x)$  exists for any  $x \in U_{\delta_0}(S \cap E)$ ,  $|t| \leq 100\delta_0$ . Let  $L_0 = \cup_{t \leq \delta_0} \xi_t(\mathcal{L})$ . One has

**Lemma 3.1**  $L_0$  is a Lagrangian submanifold in  $(R^{4n}, \omega)$ .

Proof. Let

$$\begin{aligned} F : [-\delta_0, \delta_0] \times \mathcal{L} &\rightarrow \bar{Q}_\delta \\ F(t, l) &= \xi_t(l). \end{aligned} \tag{3.1}$$

Then,

$$\begin{aligned} F^* \omega &= F_t^* \omega + i_{X_S} \Omega \wedge dt \\ &= 0 - (ds|_S) \wedge dt = 0 \end{aligned} \tag{3.2}$$

This checks that  $L_0 = F([-\delta_0, \delta_0]) \times \mathcal{L}$  is Lagrangian submanifold.

**Lemma 3.2** Let  $M = E \cap S$  and  $\omega_M = \omega|_M$ . Then there exist  $\delta_0 > 0$  and a neighbourhood  $M_0$  of  $\mathcal{L}$  in  $M$  such that  $G : M_0 \times [-\delta_0, \delta_0] \times [-\delta_0, \delta_0] \rightarrow R^{4n}$  defined by  $G(m, t, s) = \eta_s(\xi_t(m))$  is an symplectic embedding.

$$G^* \omega = \omega_M + dt \wedge ds \tag{3.3}$$

Proof. It is obvious.

Let  $U_T = G(M_T \times [-\delta_0, \delta_0] \times [-T, T])$ . If there does not exist periodic solution in  $Q_\delta(\Sigma)$  and  $M_T$  is a very small neighbourhood of  $\mathcal{L}$ , then  $s(x)$

and  $X_S$  is well defined on  $U_T$ . Therefore, there exists  $\delta_T$  such that the flow  $\xi_t(x)$  of  $X_S$  exists for any  $x \in U_T$ ,  $|t| \leq 100\delta_T$ . Let  $U_0 = U_{\delta_0}(S \cap E) \cap U_T$ ,  $U_k = \eta_{k\delta_0}(U_0) \subset U_T$ ,  $k = 1, \dots, k_T$ . Let  $X_k = X_S|_{U_k}$ . Let  $\bar{X}_k = \eta_{k\delta_0*}X_0$ . We claim that  $\bar{X}_k = X_k$ . Since  $s(x)$  and  $H(x)$  is defined on  $U_T$  and  $\{H, s\} = 1$ , so  $\xi_t(\eta_s(x)) = \eta_s(\xi_t(x))$  for  $x \in U_T$ . Differentiate it, we get  $X_S(\eta_s(x)) = \eta_{s*}X_S(x)$ . Take  $s = k$ , one proves  $\bar{X}_k = X_k$ . Recall that the flow  $\xi_t(x)$  of  $X_S$  exists for any  $x \in U_{\delta_0}(S \cap E)$ ,  $|t| \leq 100\delta_0$ . So, the flow  $\bar{\xi}_t^k(x)$  of  $\bar{X}_k$  exists for any  $x \in U_k$ ,  $|t| \leq 100\delta_0$ . Therefore, the flow  $\xi_t^k(x)$  of  $X_k$  exists for any  $x \in U_k$ ,  $|t| \leq 100\delta_0$ . This Proves that the flow  $\xi_t(x)$  of  $X_S$  exists for any  $x \in U_T$ ,  $|t| \leq 100\delta_0$ .

**Theorem 3.1** (*Long Darboux theorem*) *Let  $M = E \cap S$ ,  $\omega_M = \omega|_M$ . Let  $M_0$  as in Lemma 3.2. Let  $(U'_T, \omega') = (M_T \times [-\delta_0, \delta_0] \times [-T, T], \omega_M + dH' \wedge ds')$ . If there does not exist periodic solution in  $Q_\delta(\Sigma)$ , then there exists a symplectic embedding  $G : U'_T \rightarrow \bar{Q}_\delta$  defined by  $G(m, H', s') = \eta_{s'}(\xi_{H'}(m))$  such that*

$$G^*\omega = \omega_M + dH' \wedge ds'. \quad (3.4)$$

Proof. We follow the Arnold's proof on Darboux's theorem in [1]. Take a Darboux chart  $U$  on  $M$ , we assume that  $\omega_M|_U = \sum_{i=1}^{2n-1} dp'_i \wedge dq'_i$ . Now computing the Poisson brackets  $\{, \}^*$  of  $(p'_1, q'_1; \dots, p'_{2n-1}, q'_{2n-1}; H', s')$  for  $G^*\omega$  on  $U'_T$ . Let  $P'_i(G(p'_1, q'_1; \dots, p'_{2n-1}, q'_{2n-1}; H', s')) = P'_i(\xi_{H'}\eta_{s'}(p', q')) = p'_i$ , and  $Q'_i(G(p'_1, q'_1; \dots, p'_{2n-1}, q'_{2n-1}; H', s')) = Q'_i(\xi_{H'}\eta_{s'}(p', q')) = q'_i$ ,  $i = 1, \dots, 2n-1$ .  $\{H', s'\}^* = G^*\omega(\frac{\partial}{\partial H'}, \frac{\partial}{\partial s'}) = \omega(G_*\frac{\partial}{\partial H'}, G_*\frac{\partial}{\partial s'}) = \omega(X_H, X_S) = 1$ ,  $\{H', p'_i\}^* = G^*\omega(\frac{\partial}{\partial H'}, \frac{\partial}{\partial p'_i}) = \omega(G_*\frac{\partial}{\partial H'}, G_*\frac{\partial}{\partial p'_i}) = \omega(X_H, X_{P'_i}) = -X_{P'_i}(H) = 0$ . Similarly,  $\{H', q'_i\}^* = 0$ . Similarly,  $\{s', H'\}^* = 1$ ,  $\{s', p'_i\}^* = 0$ ,  $\{s', q'_i\}^* = 0$ . Note that  $\omega = \xi_t^*\eta_s^*\omega$ , so  $\{p'_i, q'_j\}^* = G^*\omega(\frac{\partial}{\partial p'_i}, \frac{\partial}{\partial q'_j}) = \omega(G_*\frac{\partial}{\partial p'_i}, G_*\frac{\partial}{\partial q'_j}) = \omega(\xi_{H'}\eta_{s'}*\frac{\partial}{\partial p'_i}, \xi_{H'}\eta_{s'}*\frac{\partial}{\partial q'_j}) = \xi_{H'}^*\eta_{s'}^*\omega(\frac{\partial}{\partial p'_i}, \frac{\partial}{\partial q'_j}) = \omega(\frac{\partial}{\partial p'_i}, \frac{\partial}{\partial q'_j}) = \omega_M(\frac{\partial}{\partial p'_i}, \frac{\partial}{\partial q'_j}) = \delta_{ij}$ . This shows that the Poisson brackets  $\{, \}^*$  is same as the Poisson brackets  $\{, \}'$  for  $\omega'$ . So,  $\omega' = G^*\omega$ . This finishes the proof.

Take a disk  $M_0$  enclosed by the circle  $E_0$  which is parametrized by  $t \in [0, \delta_0]$  in  $(s', H')$  - plane such that  $M_0 \subset [-2s_0, 2s_0] \times [0, \varepsilon]$  and  $\text{area}(M_0) \geq 2s_0\varepsilon$ . Now one checks that  $L = G(\mathcal{L} \times E_0)$  satisfy

$$\omega|_L = G^*\omega(\mathcal{L} \times E_0) = dH' \wedge ds'|_{E_0} = 0. \quad (3.5)$$

So,  $L$  is a Lagrangian submanifold.

**Lemma 3.3** *If there does not exist any periodic orbit in  $(Q_\delta, X_H)$ , then  $L$  is a close Lagrangian submanifold. Moreover*

$$l(V, L, \omega) = \text{area}(M_0) \quad (3.6)$$

Proof. It is obvious that  $F$  is a Lagrangian embedding. If the circle  $C$  homotopic to  $C_1 \subset \mathcal{L} \times s_0$  then we compute

$$\int_C F^*(p_i dq_i) = \int_{C_1} F^*(p_i dq_i) = 0. \quad (3.7)$$

since  $\lambda|_{C_1} = 0$  due to  $C_1 \subset \mathcal{L}$  and  $\mathcal{L}$  is “Legendre submanifold”.

If the circle  $C$  homotopic to  $C_1 \subset l_0 \times S^1$  then we compute

$$\int_C F^*(p_i dq_i) = \int_{C_1} F^*(p_i dq_i) = n(\text{area}(M_0)). \quad (3.8)$$

This proves the Lemma.

### 3.2 Second case: single curve of periodic orbits

**Theorem 3.2** *(Long Darboux cover theorem) Let  $M_0$  be as in Lemma 3.2. Let  $\bar{U}'_T = M_0 \times [-\delta_0, \delta_0] \times [-T, T]$ ,  $\omega' = \omega_M + dH' \wedge ds'$ . Then there exists a symplectic immersion  $G : U'_T \rightarrow \bar{Q}_\delta$  satisfies  $G(m; H', s') = \xi_{H'} \eta_{s'}(m)$ ; and*

$$G^* \omega = \omega_M + dH' \wedge ds'. \quad (3.9)$$

Proof. By the proof of Theorem 3.1.

Now Assume that  $H$  is “single”, i.e., there exist only one family of periodic orbits, more precicely, the periodic orbits consist of  $x(t, (\sigma_0, c))$  for some  $0 < c < \delta$  in  $Q_\delta$  such that  $H(x(t, (\sigma, c))) = c$  with period  $T_c$ . Now we assume that there does not exist any peiodic orbit on  $H^{-1}(0)$ . Let  $Z'_T = \mathcal{L} \times [-\delta_0, \delta_0] \times [-T, T]$ .  $Z_T = G(Z'_T)$ . Let  $\gamma(l) = L_0 \cap (Z_T \setminus L_0)$  and  $\{\gamma'_i\}_{i=1}^m = G^{-1}(\gamma) \subset Z'_T$ . We claim that one still can take a disk  $M'_0$  enclosed by the circle  $E'_0$  which is parametrized by  $t \in [0, t_0]$  in  $(s', H')$  – plane such that  $G|_{E'_0}$  is an embedding and  $M'_0 \subset [-2s_0, 2s_0] \times [0, \varepsilon]$  with  $0 < \varepsilon < \delta_0$  and  $\text{area}(M'_0) \geq 2s_0\varepsilon$ . In fact one can draw a curve  $E'_1$  like rectangle without bottom under the level  $H' = \varepsilon$  above  $s'$  – axis between  $\gamma'_1$  and  $\gamma'_2$  on  $(s', H')$ –plane. Let  $E_1 = G(E'_1)$  and  $\{E'_i\}_{i=1}^n = G^{-1}(E_1)$ , one can draw similar

graph curve  $F'_2$  over  $s' - axis$  below  $E'_2$  between  $\gamma'_2$  and  $\gamma'_3$  on  $(s', H')$ -plane. We do this similarly  $n$  times. Then, we connect  $E'_1, F'_2, \dots$ , etc. below  $\gamma'_i$  above  $s' - axis$  to get a graph curve  $\Gamma$ . Finally, we close  $\Gamma$  with  $s' - axis$  to get  $E'_0$ . Let  $L = G(\mathcal{L} \times E'_0)$ . So,  $L$  is again a Lagrangian submanifold. The Lemma 3.2 still holds in this case.

### 3.3 Gromov's figure eight construction

First we note that the construction of section 3.1 holds for any symplectic manifold. Now let  $(M, \omega)$  be an exact symplectic manifold with  $\omega = d\alpha$ . Let  $\Sigma = H^{-1}(0)$  be a regular and close smooth hypersurface in  $M$  and  $H$  is  $T - finite$ .  $H$  is a time-independent Hamilton function. Set  $(V', \omega') = (M \times M, \omega \ominus \omega)$ . If there does not exist any close orbit for  $X_H$  in  $(\Sigma, X_H)$ , one can construct the Lagrangian submanifold  $L$  as in section 3.1, let  $W' = L$ . Let  $h_t = h(t, \cdot) : M \rightarrow M$ ,  $0 \leq t \leq 1$  be a Hamiltonian isotopy of  $M$  induced by hamilton fuction  $H_t$  such that  $h_1(\Sigma) \cap \Sigma = \emptyset$ ,  $|H_t| \leq C_0$ . Let  $\bar{h}_t = (id, h_t)$ . Then  $F'_t = \bar{h}_t : W' \rightarrow V'$  be an isotopy of Lagrangian embeddings. As in [8], we can use symplectic figure eight trick invented by Gromov to construct a Lagrangian submanifold  $W$  in  $V = V' \times R^2$  through the Lagrange isotopy  $F'$  in  $V'$ , i.e., we have

**Proposition 3.1** *Let  $V', W'$  and  $F'$  as above. Then there exists a weakly exact Lagrangian embedding  $F : W' \times S^1 \rightarrow V' \times R^2$  with  $W = F(W' \times S^1)$  is contained in  $M \times M \times B_R(0)$ , here  $4\pi R^2 = 8C_0$  and*

$$l(V', W, \omega) = area(M'_0) = A(T). \quad (3.10)$$

Proof. Similar to [8, 2.3B'\_3].

**Example.** Let  $M$  be an open manifold and  $(T^*M, p_i dq_i)$  be the cotangent bundle of open manifold with the Liouville form  $p_i dq_i$ . Since  $M$  is open, there exists a function  $g : M \rightarrow R$  without critical point. The translation by  $tTdg$  along the fibre gives a hamilton isotopy of  $T^*M : h_t^T(q, p) = (q, p + tTdg(q))$ , so for any given compact set  $K \subset T^*M$ , there exists  $T = T_K$  such that  $h_1^T(K) \cap K = \emptyset$ .

## 4 Proof on Theorems

Take  $T_0 > 0$  such that  $A(T_0) \geq 100\pi r_0^2$ . Assume that on  $H^{-1}(0)$  there does not exist periodic orbit with period  $T \leq 100T_0$  and  $H$  as in section 3, then by the results in section 3, we have a close Lagrangian submanifold  $W = L$  or  $W = F(W' \times S^1)$  contained in  $V = M \times C \times M \times B_{r_0}(0)$  or  $V = M \times M \times B_{r_0}(0)$ . By Lagrangian non-squeezing theorem, i.e., Theorem 2.1, we have

$$A(T_0) \leq \text{area}(M_0) = l(V, W, \omega) \leq 2\pi r_0^2. \quad (4.1)$$

This is a contradiction. This contradiction shows that there is a periodic orbit with period  $T \leq 100T_0$ . This completes the proofs of theorems.

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